

STRINGY PRODUCT ON TWISTED ORBIFOLD K-THEORY FOR ABELIAN QUOTIENTS

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ABSTRACT. In this paper we present a model to calculate the stringy product on twisted orbifold K-theory of Adem-Ruan-Zhang for abelian complex orbifolds.

In the first part we consider the non-twisted case on an orbifold presented as the quotient of a manifold acted by a compact abelian Lie group. We give an explicit description of the obstruction bundle, we explain the relation with the product defined by Jarvis-Kaufmann-Kimura and, via a Chern character map, with the Chen-Ruan cohomology, and we explicitly calculate the stringy product for a weighted projective orbifold.

In the second part we consider orbifolds presented as the quotient of a manifold acted by a finite abelian group and twistings coming from the group cohomology. We show a decomposition formula for twisted orbifold K-theory that is suited to calculate the stringy product and we use this formula to calculate two examples when the group is $(\mathbb{Z}/2)^3$.

1. INTRODUCTION

This paper is devoted to the study of the stringy product on the twisted orbifold K-theory defined by Adem-Ruan-Zhang [ARZ].

In the first part we consider the non-twisted case on an orbifold presented as the quotient of a manifold acted by a compact abelian Lie group. By means of methods used to understand the obstruction bundle developed in [CH06, JKK07], we were able to give a very explicit description of the obstruction bundle in terms of G -equivariant bundles. With this model in hand, the calculation of the obstruction bundle becomes a simple diagonalization procedure and a projection onto some of the components. One of the key advantages of this construction is that everything can be performed without the use of rational coefficients, and therefore the torsion classes of equivariant K-theory are not disregarded.

We construct a stringy ring on the K-theory of the Borel construction of the inertia orbifold, similar to the stringy structure in K-theory that was defined in [JKK07]. This construction does not need rational coefficients either, and uses the same principle of the stringy product.

Applying a calibrated Chern character map (mainly due to Jarvis-Kaufmann-Kimura [JKK07]), we were able to prove that there is a ring homomorphism between

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the stringy ring on the K-theory of the inertia orbifold and the Chen-Ruan cohomology of the orbifold. We come to the conclusion that the right place to study stringy products is K-theory and not cohomology, this due to the fact that the torsion classes in the stringy K-theory are present.

We calculate an explicit example of the rings that we have defined in the case of a weighted projective orbifold, and for doing so, we make use of a decomposition of equivariant K-theory developed in [AR03]

In the second part we consider orbifolds presented as the quotient of a manifold acted by a finite abelian group and twistings coming from the group cohomology. We recall the definition of the twisting classes via the inverse transgression map and we give a simple argument to show that these classes must be torsion. We calculate the inverse transgression map in the case that the group is $(\mathbb{Z}/2)^3$ and we expand on an example previously constructed in [ARZ].

Then we show a decomposition formula for twisted orbifold K-theory that can be considered as an hybrid between the decomposition formula for equivariant K-theory using cyclotomic fields and the decomposition for twisted orbifold K-theory (both in [AR03]). We finish the paper using this formula to calculate the stringy product in twisted orbifold K-theory with rational coefficients when the orbifolds are $[\ast/(\mathbb{Z}/2)^3]$ and $[T^6/(\mathbb{Z}/2)^3]$, and the twisting classes are non-trivial.

It was pointed out to us by A. Adem that the decomposition formula is not necessary to calculate the stringy product in twisted orbifold K-theory in the case of $[\ast/(\mathbb{Z}/2)^3]$ and an explicit calculation of this ring was carried out by A. N. Duman in [Dum].

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2. STRINGY PRODUCT ON THE K-THEORY OF THE INERTIA ORBIFOLD

Let M be a compact manifold and G a compact abelian Lie group that acts almost freely on M , such that the orbifold $[M/G]$ is endowed with an almost complex structure. In what follows we will define a stringy product on the equivariant K-theory of the inertia orbifold of $[M/G]$ in as much the same way as the Chen-Ruan product in orbifold cohomology. We will show that our definition will be equivalent to the one defined in [ARZ] without twisting. The key ingredient of the definition will be the obstruction bundle and its construction will be outlined in this section.

Consider the set $\mathcal{C} = \{g \in G \mid M^g \neq \emptyset\}$ where M^g denotes the fixed point set of the action of g . As G is compact abelian and the action is almost free, the set \mathcal{C} is finite.

In the case that we are interested, the inertia orbifold (see [Moe02]) of $[M/G]$ is

$$I[M/G] = \sqcup_{g \in \mathcal{C}} [M^g/G]$$

where G acts in the natural way.

Definition 2.1. The stringy K-theory of the inertia orbifold $K_{st}([M/G])$, as a \mathbb{Z} -module, is defined as the K-theory of its inertia orbifold, i.e.

$$K_{st}^*([M/G]) := K^*(I[M/G]) \cong \bigoplus_{g \in \mathcal{C}} K_G^*(M^g) \times \{g\}.$$

Note that the summands in the right hand side are in G -equivariant K-theory [Seg68].

The ring structure of the stringy K-theory defined in [ARZ] uses the genus zero, three point function, Gromov-Witten invariants approach. In this paper we will show an equivalent description for this ring structure that has the advantage that it is very simple to use when one wants to calculate explicit examples.

2.1. Obstruction bundle. Let us start by noting that as the orbifold $[M/G]$ is almost complex, then the normal bundles of the fixed point sets M^g are all G -equivariant complex bundles. Now, take $g, h \in \mathcal{C}$ and consider the space $M^{g,h} := M^g \cap M^h$. As g and h commute, then one can simultaneously diagonalize the action of g and h on the normal bundle $V_{g,h}$ of the inclusion $M^{g,h} \rightarrow M$. Therefore, as the order of both g and h is finite, $V_{g,h}$ is a complex bundle isomorphic to $V_1 \oplus \cdots \oplus V_k$ where V_j is simultaneously the eigen bundle of $e^{2\pi i a_j}$ for the action of g and of $e^{2\pi i b_j}$ for the action of h , with $a_j, b_j \in \mathbb{Q}$ and $0 \leq a_j, b_j < 1$. Note that (and this is a key fact) as the group G is abelian, then all the bundles V_j are G -equivariant (this construction is performed in each connected component of $M^{g,h}$).

Definition 2.2. The obstruction bundle $D_{g,h}$ for the pair g, h is defined as:

$$D_{g,h} := \bigoplus_{\{j \mid a_j + b_j > 1\}} V_j,$$

and is a G -equivariant bundle over $M^{g,h}$.

Remark 2.3. If we take $k = (gh)^{-1}$ and $e^{2\pi i c_j}$ the eigenvalue of the action of k in V_j ($0 \leq c_j < 1$), then we have also that

$$D_{g,h} = \bigoplus_{\{j \mid a_j + b_j + c_j = 2\}} V_j.$$

This bundle is the obstruction bundle that is given in [CH06, Prop. 1 page 65] for abelian orbifolds. We are adding the fact that this bundle is G -equivariant.

The obstruction bundle $D_{g,h}$ defined above is equivalent to the obstruction bundle given in [JKK07, Def. 1.5] in the case that the group G is finite. Let's recall their definition.

For $g \in \mathcal{C}$ split the normal bundle W_g of the inclusion $M^g \rightarrow M$ into eigen bundles of the g action. Let W_g^l be the eigen bundle of $e^{2\pi i a_l}$ and define the bundle (in $K^0(M^g) \otimes \mathbb{Q}$)

$$(2.1) \quad \mathcal{F}_g := \bigoplus_l a_l W_g^l.$$

The obstruction bundle $B_{g,h}$ of [JKK07] is

$$(2.2) \quad B_{g,h} := TM^{g,h} \ominus TM|_{M^{g,h}} \oplus \mathcal{F}_g|_{M^{g,h}} \oplus \mathcal{F}_h|_{M^{g,h}} \oplus \mathcal{F}_{(gh)^{-1}}|_{M^{g,h}}.$$

Lemma 2.4. When the group G is finite, the bundle $D_{g,h}$ and the virtual bundle $B_{g,h}$ represent the same class in $K^0(M^{g,h}) \otimes \mathbb{Q}$.

Proof. Let's write the virtual bundle $B_{g,h}$ in terms of the bundles V_j . By restriction we have that

$$\mathcal{F}_g|_{M^{g,h}} \cong \bigoplus_j a_j V_j, \quad \mathcal{F}_h|_{M^{g,h}} \cong \bigoplus_j b_j V_j \quad \text{and} \quad \mathcal{F}_{(gh)^{-1}}|_{M^{g,h}} \cong \bigoplus_j c_j V_j.$$

Then

$$B_{g,h} \cong \bigoplus_j a_j V_j \oplus \bigoplus_j b_j V_j \oplus \bigoplus_j c_j V_j \ominus \bigoplus_j V_j \cong \bigoplus_j (a_j + b_j + c_j - 1) V_j = D_{g,h}.$$

□

2.2. Stringy Product on K-theory of the inertia orbifold. With the obstruction bundle in hand, we can define now the stringy product on the K-theory of the inertia orbifold. Let's denote the inclusions by

$$e_1 : M^{g,h} \rightarrow M^g \quad e_2 : M^{g,h} \rightarrow M^h \quad e_3 : M^{g,h} \rightarrow M^{gh}.$$

Definition 2.5. For the almost complex orbifold $[M/G]$, define the stringy product \star in the stringy K-theory $K_{st}^*([M/G])$ as follows:

$$\begin{aligned} \star : (K_G^*(M^g) \times \{g\}) \times (K_G^*(M^h) \times \{h\}) &\rightarrow K_G^*(M^{gh}) \times \{gh\} \\ ((F, g), (H, h)) &\mapsto (e_{3!}(e_1^* F \otimes e_2^* H \otimes \lambda_{-1}(D_{g,h})), gh), \end{aligned}$$

where $g, h \in \mathcal{C}$, the pullbacks and pushforwards are defined in the equivariant setup and $\lambda_{-1}(D_{g,h})$ is the Euler class of the bundle $D_{g,h}$, i.e. the restriction to $M^{g,h}$ of the Thom class of $D_{g,h}$ (see [Seg68, Prop. 3.2]).

Proposition 2.6. *The ring structure in $K_{st}^*([M/G])$ defined above in definition 2.5 is the same as the ring structure defined in Definition 7.3 of [ARZ].*

Proof. By remark 2.3 above we know that in Proposition 1 of [CH06] it is proved that the obstruction bundle of the Chen-Ruan cohomology is isomorphic to the obstruction bundle $D_{g,h}$. As in [ARZ] the obstruction bundle is the obstruction bundle of the Chen-Ruan cohomology [CR04], then the two ring structures agree. □

2.3. Associativity. The fact that $K_{st}^*([M/G])$ together with the stringy product in K-theory is associative is proved in Theorem 7.4 [ARZ]. It is basically the same proof that is given in lemma 5.4 of [JKK07] but in the equivariant setup. We are not going to reproduce the proof in here because it would be redundant. Instead, we will write some words on the key ingredient of the proof which is the equivariant version of the excess intersection formula of Quillen.

In Proposition 3.3 of [Qui71] it is shown that for the inclusion of almost complex manifolds

$$\begin{array}{ccc} S_1 & \xrightarrow{i_1} & S \\ j_1 \uparrow & & \uparrow i_2 \\ V & \xrightarrow{j_2} & S_2 \end{array}$$

with $V = S_1 \cap S_2$, then

$$i_2^* i_{1*} x = j_{2*}(e(E(S, S_1, S_2)) \cdot j_1^* x)$$

holds in cohomology where $e(E(S, S_1, S_2))$ is the Euler class of the bundle

$$E(S, S_1, S_2) = TS|_V \oplus TV \ominus TS_1|_V \ominus TS_2|_V.$$

This bundle is known as the excess intersection bundle.

The formula also holds in K-theory, namely

$$i_2^* i_{1!} F = j_{2!} (\lambda_{-1}(E(S, S_1, S_2)) \otimes j_1^* F).$$

Both formulas rely on the fact that for a complex bundle $E \rightarrow V$ with zero section $\sigma : V \rightarrow E$, one has $\sigma^* \sigma_* x = e(E) \cdot x$ in cohomology and $\sigma^* \sigma_! F = \lambda_{-1}(E) \otimes F$ in K-theory.

In the case that the bundle $E \rightarrow V$ is equivariant, Segal showed that the formula $\sigma^* \sigma_! F = \lambda_{-1}(E) \otimes F$ also applies to the equivariant setup [Seg68, Page 140]; therefore the excess intersection formula of Quillen holds in the case that the manifolds S, S_1, S_2, V and the maps i_i, j_1, i_2, j_2 are all equivariant.

With the equivariant excess intersection formula in hand, the associativity boils down to check the following equality in $K_G^0(N)$ where $N = M^{g,h,k}$:

$$D_{g,h}|_N \oplus D_{g,h,k}|_N \oplus E(M^{g,h}, M^{g,h,k}, M^{g,h,k}) = D_{g,h,k}|_N \oplus D_{h,k}|_N \oplus E(M^{h,k}, M^{g,h,k}, M^{h,k}).$$

The proof follows the same lines as the proof of lemma 5.4 of [JKK07]; we only need to remark that in our case all the operations are done in the G -equivariant setup. We will not reproduce the proof.

2.4. Weighted projective orbifold $\mathbb{CP}[p : 1 : \dots : 1]$. In this section we are going to calculate the stringy K-theory in the case of the weighted projective orbifold $\mathbb{CP}[p : 1 : \dots : 1]$. Let's recall its definition.

Let $M = S^{2n+1} = \{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} \mid \sum_i |z_i|^2 = 1\}$ and $G = S^1$ acting on M as

$$(2.3) \quad \begin{aligned} S^1 \times S^{2n+1} &\rightarrow S^{2n+1} \\ (\lambda, (z_0, \dots, z_n)) &\mapsto (\lambda^p z_0, \lambda z_1, \lambda z_2, \dots, \lambda z_n). \end{aligned}$$

with p a prime number. The quotient space M/G is the weighted projective space $\mathbb{CP}(p, 1, \dots, 1)$ and the orbifold $[M/G]$ is the weighted projective orbifold $\mathbb{CP}[p : 1 : \dots : 1]$.

The set \mathcal{C} is equal to the elements in the group $\mathbb{Z}/p \subset G$ which we will denote by $\{g_0 = id, g_1, \dots, g_{p-1}\} = \mathcal{C}$. The fixed point sets are $M^{g_0} = M$ and $M^{g_j} = \{(z_0, 0, \dots, 0) \in M\} = S^1$ for $1 \leq j < p$. The G action on M^{g_j} is the same as the action of G on the space $S^1/(\mathbb{Z}/p)$ given by rotation. Then as a \mathbb{Z} -module

$$K_{st}^*([M/G]) = K_G^*(S^{2n+1}) \times \{g_0\} \oplus \bigoplus_{j=1}^{p-1} K_G^*(S^1/(\mathbb{Z}/p)) \times \{g_j\}.$$

As the action of G in $M^{g_j} \cong S^1/(\mathbb{Z}/p)$ is by rotation, then we know that

$$K_G^*(S^1/(\mathbb{Z}/p)) \cong K_{\mathbb{Z}/p}^*(pt) \cong R(\mathbb{Z}/p),$$

where $R(\mathbb{Z}/p)$ is the Grothendieck ring of \mathbb{Z}/p representations. Therefore, the stringy K-theory, as a \mathbb{Z} -module is isomorphic to

$$K_{st}^*([M/G]) = K_G^*(S^{2n+1}) \times \{g_0\} \oplus \bigoplus_{j=1}^{p-1} R(\mathbb{Z}/p) \times \{g_j\}.$$

For $g_j, g_k \in \mathcal{C}$, $j \neq 0$, the obstruction bundle $D_{j,k} := D_{g_j, g_k}$ is a subbundle of the normal bundle $V_{j,k}$ of the inclusion $S^1 \rightarrow S^{2n+1}$, where in this case $M^{g_j, g_k} = M^{g_j} = S^1$. The normal bundle $V_{j,k}$ is isomorphic to the trivial bundle $S^1 \times \mathbb{C}^n$

over S^1 , where the circle action is given by $\lambda(\eta, \mathbf{z}) = (\lambda^p \eta, \lambda \mathbf{z})$. This follows from the fact that one can take the set

$$\{(z_0, \dots, z_n) \in S^{2n+1} \mid |z_i| < \frac{1}{n} \text{ for } 1 \leq i \leq n\}$$

as tubular neighborhood of the inclusion $S^1 \rightarrow S^{2n+1}$, and this set is isomorphic to

$$S^1 \times \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid |z_i| < \frac{1}{n}\}.$$

Then the normal bundle can be seen as $nW \in R(\mathbb{Z}/p) \cong K_{S^1}^*(M^{g_j, g_k})$ where W is the one dimensional irreducible representation whose character is $e^{\frac{2\pi i}{p}}$.

Going back to the obstruction bundle, we have that

$$(2.4) \quad D_{j,k} = \begin{cases} V_{j,k} & \text{for } j+k > p \\ 0 & \text{for } j+k \leq p; \end{cases}$$

this is because the action of g_j on the normal bundle $V_{j,k}$ is given by multiplying by $e^{\frac{2\pi i}{p}j}$.

Having defined the obstruction bundle, we need to explicitly determine the pull-backs and pushforwards of the inclusion maps. The first step is to calculate explicitly $K_G^*(M)$, and for this we are going to use a decomposition theorem for orbifold K-theory due to Adem and Ruan [AR03] (cf. [LO01])

Theorem 2.7. [AR03, Th. 5.1] *Let $[M/G]$ be a compact orbifold where G is a compact Lie group acting almost freely on M . Then there is a rational isomorphism of rings*

$$K_G^*(M) \otimes \mathbb{Q} \cong \prod_{\{(C) \mid C \subset G \text{ cyclic}\}} [K^*(M^C/Z_G(C)) \otimes \mathbb{Q}(\zeta_{|C|})]^{W_G(C)}$$

where (C) ranges over conjugacy classes of cyclic subgroups, $W_G(C) = N_G(C)/Z_G(C)$ and $\mathbb{Q}(\zeta_{|C|})$ is the cyclotomic field where $\zeta_{|C|}$ is a primitive $|C|$ -th root of unity.

For a cyclic group C , the map from left to right is the composition of maps

$$\begin{aligned} K_G^*(M) &\xrightarrow{\phi} K_{N_G(C)}^*(M^C) \xrightarrow{\psi} K_{Z_G(C)}^*(M^C) \otimes R(C) \longrightarrow \\ &\xrightarrow{\alpha} K^*(EG \times_{Z_G(C)} M^C) \otimes R(C) \xrightarrow{\beta} K^*(M^C/Z_G(C)) \otimes \mathbb{Q}(\zeta_{|C|}) \end{aligned}$$

where: ϕ is the restriction map; ψ is defined such that for any $N_G(C)$ vector bundle $E \rightarrow M^C$

$$\psi([E]) = \sum_{V \in \text{Irr}(C)} [\text{Hom}_C(V, E)] \otimes [V];$$

α is the natural map from equivariant K-theory to the K-theory of the Borel construction; and β is the map obtained from the isomorphism

$$K^*(EG \times_{Z_G(C)} M^C) \otimes \mathbb{Q} \cong K^*(M^C/Z_G(C)) \otimes \mathbb{Q}$$

that comes from the rational acyclicity of the fibers of the map $EG \times_{Z_G(C)} M^C \rightarrow M^C/Z_G(C)$, and the ring map $R(C) \otimes \mathbb{Q} \rightarrow \mathbb{Q}(\zeta_{|C|})$ whose kernel is the ideal of elements whose characters vanish in all generators of C .

Remark 2.8. We have modified slightly the statement of Theorem 5.1 in [AR03] by leaving the decomposition formula only in terms of K-theory. Applying the Chern character isomorphism

$$Ch : K^*(M^C/Z_{G(C)}) \otimes \mathbb{Q} \xrightarrow{\cong} H^*(M^C/Z_{G(C)}) \otimes \mathbb{Q}$$

one recovers the original statement.

Going back to the weighted projective orbifold M/G , we have

Lemma 2.9. *Let $G = S^1$ act on $M = S^{2n+1}$ as in (2.3), then there is an isomorphism of rings*

$$K_G^*(M) \otimes \mathbb{Q} \cong \mathbb{Q}[u]/\langle (1-u)^{n+1} \rangle \times \mathbb{Q}(\zeta)$$

where $\zeta = e^{\frac{2\pi i}{p}}$.

Proof. From theorem 2.7 we have that

$$K_G^*(M) \otimes \mathbb{Q} \cong (K^*(M/G) \otimes \mathbb{Q}) \times (K^*(M^{\mathbb{Z}/p}/G) \otimes \mathbb{Q}(\zeta))$$

because the only cyclic subgroups of the circle that have non empty fixed point set are the trivial subgroup and \mathbb{Z}/p . The quotient spaces are: $M/G = \mathbb{CP}(p : 1 : \dots : 1)$ and $M^{\mathbb{Z}/p}/G = \text{point}$, therefore

$$K_G^*(M) \otimes \mathbb{Q} \cong (K^*(\mathbb{CP}(p : 1 : \dots : 1)) \otimes \mathbb{Q}) \times \mathbb{Q}(\zeta).$$

We are left to calculate the K-theory of the weighted projective space $\mathbb{CP}_w^n := \mathbb{CP}(p : 1 : \dots : 1)$; this we will do by calculating its cohomology.

The space \mathbb{CP}_w^n has only one orbifold point whose neighborhood looks like the quotient of \mathbb{C}^n by the diagonal action of \mathbb{Z}/p . Then the space \mathbb{CP}_w^n can be thought as a smooth manifold because the orbifold point can be removed.

If we consider the inclusion $j : \mathbb{CP}^{n-1} \rightarrow \mathbb{CP}_w^n$ that skips the first variable, then we get the following short exact sequence:

$$0 \longrightarrow H^{2n}(\mathbb{CP}_w^n, \mathbb{CP}^{n-1}) \longrightarrow H^*(\mathbb{CP}_w^n) \xrightarrow{j^*} H^*(\mathbb{CP}^{n-1}) \longrightarrow 0.$$

As $H^*(\mathbb{CP}^{n-1}) = \mathbb{Z}[y]/\langle y^n \rangle$ then there must exist $x \in H^2(\mathbb{CP}_w^n)$ with $j^*x = y$. As the transversal intersection of the submanifold $j(\mathbb{CP}^{n-1})$ with itself $n-1$ times inside \mathbb{CP}_w^n gives p points (this is because the normal bundle of the inclusion j is isomorphic to the p -th power of the canonical line bundle over \mathbb{CP}^{n-1}), then x^n must be p -times the image of the generator of $H^{2n}(\mathbb{CP}_w^n, \mathbb{CP}^{n-1})$.

Therefore we can conclude that

$$H^*(\mathbb{CP}_w^n) \cong \mathbb{Z}[x, v]/\langle x^n - pv, x^{n+1}, v^2 \rangle,$$

with rational coefficients and taking $z := p^{-\frac{1}{n}}x$

$$H^*(\mathbb{CP}_w^n, \mathbb{Q}) \cong \mathbb{Q}[z]/\langle z^{n+1} \rangle,$$

and by the Chern character isomorphism

$$K^*(\mathbb{CP}_w^n) \otimes \mathbb{Q} \cong \mathbb{Q}[u]/\langle (1-u)^{n+1} \rangle$$

where $ch(1-u)^n = z^n$. □

For the fixed point set $M^{\mathbb{Z}/p}$ we already know that $K_G(M^{\mathbb{Z}/p}) \cong R(\mathbb{Z}/p) \cong \mathbb{Z}[W]/\langle W^p - 1 \rangle$. Applying again theorem 2.7 we have that

Lemma 2.10. *There is a ring isomorphism*

$$\begin{aligned} R(\mathbb{Z}/p) \otimes \mathbb{Q} &\cong \mathbb{Q} \otimes \mathbb{Q}(\zeta) \\ f(W) &\mapsto (f(1), f(\zeta)) \end{aligned}$$

where $f(W)$ is a polynomial that belongs to $\mathbb{Z}[W]/\langle W^p - 1 \rangle$ and W is the one dimensional irreducible representation of \mathbb{Z}/p whose character on $1 \in \mathbb{Z}/p$ is $e^{\frac{2\pi i}{p}}$.

Lemma 2.11. *The pullback of the inclusion map $i : M^{\mathbb{Z}/p} \rightarrow M$*

$$i^* : K_G^*(M) \otimes \mathbb{Q} \rightarrow K_G^*(M^{\mathbb{Z}/p}) \otimes \mathbb{Q}$$

is isomorphic to

$$(2.5) \quad \begin{aligned} \mathbb{Q}[u]/\langle (1-u)^{n+1} \rangle \times \mathbb{Q}(\zeta) &\rightarrow \mathbb{Q} \times \mathbb{Q}(\zeta) \\ (p(u), q(\zeta)) &\mapsto (p(1), q(\zeta)). \end{aligned}$$

Proof. It follows directly from lemmas 2.10 and 2.9. \square

The pushforward $i_!$ of the inclusion follows the same lines.

Lemma 2.12. *The pushforward*

$$i_! : K_G^*(M^{\mathbb{Z}/p}) \otimes \mathbb{Q} \rightarrow K_G^*(M) \otimes \mathbb{Q}$$

is isomorphic to

$$\begin{aligned} i_! : \mathbb{Q} \times \mathbb{Q}(\zeta) &\rightarrow \mathbb{Q}[u]/\langle (1-u)^{n+1} \rangle \times \mathbb{Q}(\zeta) \\ (r, q(\zeta)) &\mapsto (r(1-u)^n, q(\zeta)(1-\zeta)^n) \end{aligned}$$

Proof. The application in the first coordinate is the pushforward in K-theory of the inclusion of a point into the weighted projective space $\mathbb{C}P_w^n$, and the application of the second coordinate is the image in $\mathbb{Q}(\zeta)$ of the map $i^*i_! : R(\mathbb{Z}/p) \rightarrow R(\mathbb{Z}/p)$

$$i^*i_!q(W) = q(W)\lambda_{-1}(nW) = q(W)\lambda_{-1}(W)^n = q(\zeta)(1-W)^n$$

where nW represents the class of the normal bundle of the inclusion i as a G equivariant bundle. \square

We can finish this section by giving the explicit calculation of the stringy product in the orbifold K-theory of $[M/G] = \mathbb{C}P[p : 1 : \dots : 1]$.

Theorem 2.13. *The stringy ring with rational coefficients of the K-theory of the inertia orbifold of the weighted projective orbifold $[M/G] = \mathbb{C}P[p : 1 : \dots : 1]$ is*

$$K_{st}^*([M/G]) \otimes \mathbb{Q} \cong (\mathbb{Q}[u]/\langle (u-1)^{n+1} \rangle \times \mathbb{Q}(\zeta)) \times \{g_0\} \oplus \bigoplus_{j=1}^{p-1} (\mathbb{Q} \times \mathbb{Q}(\zeta)) \times \{g_j\}$$

with the product defined by the following cases:

- $(j = 0, k \neq 0)$

$$(p(u), q(\zeta), g_0) \star (r, w(\zeta), g_k) = (p(1)r, q(\zeta)w(\zeta), g_k)$$

- $(j + k < p, j \neq 0, k \neq 0)$

$$(r, q(\zeta), g_j) \star (s, w(\zeta), g_k) = (rs, q(\zeta)w(\zeta), g_{j+k})$$

- $(j + k = p, j \neq 0, k \neq 0)$

$$(r, q(\zeta), g_j) \star (s, w(\zeta), g_k) = (rs(1-u)^n, q(\zeta)w(\zeta)(1-\zeta)^n, g_0)$$

- $(j + k > p)$

$$(r, q(\zeta), g_j) \star (s, w(\zeta), g_k) = (0, q(\zeta)w(\zeta)(1 - \zeta)^n, g_{j+k-p})$$

Proof. From formula 2.4 we know that the if obstruction bundle is different than zero then it is precisely the normal bundle. Therefore the Euler class of the obstruction bundle is 1 in the case that the bundle is zero, and $(1 - \zeta)^n$ in the case that it is the normal bundle. The product structure \star follows from the definition 2.5 and of lemmas 2.9 and 2.10. \square

2.5. Relation with Chen-Ruan Cohomology. In what follows we want to prove that there is a homomorphism of rings from the stringy ring of the K-theory of the inertia orbifold to the Chen-Ruan cohomology. We will show this fact by first constructing a ring structure on the K-theory of the Borel construction of the inertia orbifold. Then, via the some modified version of the Chern character map, we will construct a ring structure on the cohomology with rational coefficients of the Borel construction of the inertia orbifold. This latter ring, once tensored with \mathbb{Q} , will be isomorphic to the Chen-Ruan cohomology.

2.6. Stringy K-theory for the Borel construction of the inertia orbifold. Let's start by reviewing the relation between equivariant K-theory and the K-theory of the Borel construction (see [AS69]).

For any G -space N with $N_G := N \times_G EG$, let's denote the map $\tau_N : K_G(N) \rightarrow K(N_G)$ that takes any G -equivariant complex vector bundle F over N and maps it to $\tau_N(F) = F_G := F \times_G EG$ a complex vector bundle over N_G . This map is a ring homomorphism and has very nice properties: it behaves well under restriction and pushforwards, namely, if we have an inclusion of G -manifolds $i : W \rightarrow N$, then

$$\tau_W i^*(L) = i^* \tau_N(L) \quad \text{and} \quad \tau_N i_!(F) = i_! \tau_W(F);$$

and it behaves well under the map λ_{-1} , i.e.

$$\tau_N(\lambda_{-1}(L)) = \lambda_{-1}(\tau_N(L)).$$

Moreover, the map τ_N realizes the famous Atiyah-Segal completion theorem [AS69].

Theorem 2.14. *Let the orbifold $[M/G]$ be as in section 2. The K-theory of the Borel construction of the inertia orbifold*

$$K_{orb}^*([M/G]) := \bigoplus_{g \in \mathcal{C}} K^*(M_G^g) \times \{g\}$$

can be endowed with the ring structure

$$\begin{aligned} \bullet : (K^*(M_G^g) \times \{g\}) \times (K^*(M_G^h) \times \{h\}) &\rightarrow K^*(M_G^{gh}) \times \{gh\} \\ ((F, g), (H, h)) &\mapsto (F \bullet H, gh). \end{aligned}$$

with $F \bullet H = e_{3!}(e_1^ F \otimes e_2^* H \otimes \lambda_{-1}(\tau_{M^{g,h}}(D_{g,h})))$.*

Proof. The associativity of the product \bullet follows from the associativity of \star (see section 2.3) and the properties of the maps τ . \square

Remark 2.15. The construction above of theorem 2.14 is due to Jarvis-Kaufmann-Kimura [JKK07] when one considers the coarse moduli space of the inertia orbifold. Ours generalizes it to the Borel construction.

Theorem 2.16. *The maps*

$$\tau_{M^g} : K_G^*(M^g) \rightarrow K^*(M_G^g)$$

for $g \in \mathcal{C}$ define a ring homomorphism

$$\tau : (K_{st}^*([M/G]), \star) \rightarrow (K_{orb}^*([M/G]), \bullet).$$

Proof. Take $F \in K_G^*(M^g)$ and $H \in K_G^*(M^h)$ and consider the following set of equalities:

$$\begin{aligned} \tau_{M^{gh}}(F \star H) &= \tau_{M^{gh}}(e_{3!}(e_1^*F \otimes e_2^*H \otimes \lambda_{-1}(D_{g,h}))) \\ &= e_{3!}(\tau_{M^{g,h}}(e_1^*F \otimes e_2^*H \otimes \lambda_{-1}(D_{g,h}))) \\ &= e_{3!}(e_1^*(\tau_{M^g}F) \otimes e_2^*(\tau_{M^h}H) \otimes \lambda_{-1}(\tau_{M^{g,h}}D_{g,h})) \\ &= \tau_{M^g}F \bullet \tau_{M^h}H. \end{aligned}$$

The theorem follows. \square

2.7. Chern Character. We can define a stringy product on the cohomology of the Borel construction of the inertia orbifold of $[M/G]$ in the same way that it was done in the previous section.

Proposition 2.17. *The cohomology of the Borel construction of the inertia orbifold*

$$H_{orb}^*([M/G]) := \bigoplus_{g \in \mathcal{C}} H^*(M_G^g, \mathbb{Z}) \times \{g\}$$

can be endowed with the ring structure

$$\begin{aligned} \circ : (H^*(M_G^g, \mathbb{Z}) \times \{g\}) \times (H^*(M_G^h, \mathbb{Z}) \times \{h\}) &\rightarrow H^*(M_G^{gh}, \mathbb{Z}) \times \{gh\} \\ ((\alpha, g), (\beta, h)) &\mapsto (\alpha \circ \beta, gh). \end{aligned}$$

with $\alpha \circ \beta = e_{3*}(e_1^*\alpha \cdot e_2^*\beta \cdot \text{eu}(\tau_{M^{g,h}}(D_{g,h})))$.

Proof. The associativity of the product \circ follows also from the associativity of product \star (see section 2.3), the properties of the map $\tau_{M^{g,h}}$ and the properties of the Euler class “eu”. \square

In order to have a well defined ring homomorphism between $(K_{orb}^*([M/G]), \bullet)$ and $(H_{orb}^*([M/G]), \circ)$ we need to calibrate the Chern character maps at the level of the spaces M_G^g with some invertible elements in cohomology. This construction was developed in [JKK07] at the level of the coarse moduli space of the inertia orbifold. The map is basically the same at the level of the Borel construction.

To construct this calibrated Chern character we need to recall some properties of the Thom isomorphisms in K-theory and in cohomology (see for instance [AS68, section 2]).

Let X be a manifold and V a complex vector bundle over X . Let

$$\phi : K^*(X) \rightarrow K^*(V)$$

$$\psi : H^*(X, \mathbb{Q}) \rightarrow H^*(V, \mathbb{Q})$$

be the Thom isomorphisms in K-theory and in cohomology respectively. Then, for any $u \in K^*(X)$ one has

$$(2.6) \quad \text{ch}(\phi(u)) = \psi(\text{ch}(u) \cdot \mu(V))$$

where the cohomology class $\mu(V)$ is defined as

$$\mu(V) := \prod \frac{1 - e^{x_i}}{x_i}$$

where the x_i 's are the Chern roots of V . Moreover, the class $\mu(V)$ is multiplicative, i.e. $\mu(V \oplus F) = \mu(V) \cdot \mu(F)$ and measures the difference of the Chern character of the Euler class in K-theory with the one in cohomology, namely

$$(2.7) \quad \text{ch}\lambda_{-1}(V) = \text{eu}(V) \cdot \mu(V).$$

If we have an inclusion of manifolds $i : X \rightarrow Y$ with normal bundle V and pushforward maps

$$\begin{aligned} i_! : K^*(X) &\rightarrow K^*(Y) \\ i_* : H^*(X, \mathbb{Q}) &\rightarrow H^*(Y, \mathbb{Q}) \end{aligned}$$

then one has the equality

$$\text{ch}(i_! u) = i_*(\text{ch}(u) \cdot \mu(V))$$

where in this case the cohomology class $\mu(V)$ has support on the normal bundle of X .

We are now ready to define the Chern character map for the orbifold K-theory.

Definition 2.18. [JKK07, Formula (55)] The orbifold Chern character is the homomorphism

$$Ch : K_{orb}^*([M/G]) \rightarrow H_{orb}^*([M/G]; \mathbb{Q})$$

such that for all $g \in \mathcal{C}$ is defined as

$$\begin{aligned} Ch : K^*(M_G^g) &\rightarrow H^*(M_G^g; \mathbb{Q}) \\ F &\mapsto \text{ch}(F)\mu^{-1}(\tau_{M^g}\mathcal{F}_g) \end{aligned}$$

where the G -bundle \mathcal{F}_g over M^g is defined in formula 2.1.

The fact that the orbifold Chern character is a ring homomorphism depends on the following lemma.

Lemma 2.19. *In $K_G(M^{g,h})$ the following equality holds*

$$D_{g,h} \oplus TM^{gh}|_{M^{g,h}} \ominus TM^{g,h} = \ominus e_1^*\mathcal{F}_g \ominus e_2^*\mathcal{F}_h \oplus e_3^*\mathcal{F}_{gh}.$$

Proof. We know that

$$\mathcal{F}_{gh} \oplus \mathcal{F}_{(gh)^{-1}} = TM|_{M^{gh}} \ominus TM^{gh}.$$

The equality then holds from the fact that $D_{g,h} = B_{g,h}$ where $B_{g,h}$ is defined in formula 2.2. \square

Applying the homomorphism $\tau_{M^{g,h}}$ and the map μ we get,

Corollary 2.20.

$$\begin{aligned} \mu(\tau_{M^{g,h}} D_{g,h}) \mu(\tau_{M^{g,h}} (TM^{gh}|_{M^{g,h}} \ominus TM^{g,h})) e_3^* \mu^{-1}(\tau_{M^{gh}} \mathcal{F}_{gh}) = \\ e_1^* \mu^{-1}(\tau_{M^g} \mathcal{F}_g) e_2^* \mu^{-1}(\tau_{M^h} \mathcal{F}_h). \end{aligned}$$

Theorem 2.21. [JKK07, Theorem 6.1] *The orbifold Chern character is a ring homomorphism*

$$Ch : (K_{orb}^*([M/G]), \bullet) \rightarrow (H_{orb}^*([M/G]; \mathbb{Q}), \circ).$$

Proof. Take $F \in K^*(M_G^g)$ and $H \in H^*(M_G^h)$ and consider the following set of equalities:

$$\begin{aligned}
Ch(F \bullet H) &= ch(F \bullet H) \mu^{-1}(\tau_{M^{gh}} \mathcal{F}_{gh}) \\
&= ch(e_3! (e_1^* F \otimes e_2^* H \otimes \lambda_{-1}(\tau_{M^{g,h}} D_{g,h}))) \mu^{-1}(\tau_{M^{gh}} \mathcal{F}_{gh}) \\
&= e_{3*} \left(e_1^* ch F \ e_2^* ch H \ eu(\tau_{M^{g,h}} D_{g,h}) \ \mu(\tau_{M^{g,h}} D_{g,h}) \right. \\
&\quad \left. \mu(\tau_{M^{g,h}} (TM^{gh}|_{M^{g,h}} \ominus TM^{g,h})) \right) \mu^{-1}(\tau_{M^{gh}} \mathcal{F}_{gh}) \\
&= e_{3*} \left(e_1^* ch F \ e_2^* ch H \ eu(\tau_{M^{g,h}} D_{g,h}) \ \mu(\tau_{M^{g,h}} D_{g,h}) \right. \\
&\quad \left. \mu(\tau_{M^{g,h}} (TM^{gh}|_{M^{g,h}} \ominus TM^{g,h})) \ e_3^* \mu^{-1}(\tau_{M^{gh}} \mathcal{F}_{gh}) \right) \\
&= e_{3*} \left(e_1^* ch F \ e_2^* ch H \ eu(\tau_{M^{g,h}} D_{g,h}) e_1^* \mu^{-1}(\tau_{M^g} \mathcal{F}_g) e_2^* \mu^{-1}(\tau_{M^h} \mathcal{F}_h) \right) \\
&= e_{3*} \left(e_1^* Ch F \ e_2^* Ch H \ eu(\tau_{M^{g,h}} D_{g,h}) \right) \\
&= Ch F \circ Ch H.
\end{aligned}$$

From the second to the third line we used formula 2.6, the fourth line is by the properties of the pushforward and the fifth line is by corollary 2.20. \square

Using the fact that the Chern character map becomes a $\mathbb{Z}/2$ graded vector space isomorphism when one tensors the K-theory with the rational numbers, and because the classes given by μ are all invertible, we can deduce that

Corollary 2.22. *The orbifold Chern character is a ring isomorphism when one tensors the orbifold K-theory with the rationals, i.e.*

$$Ch : (K_{orb}^*([M/G]) \otimes \mathbb{Q}, \bullet) \xrightarrow{\cong} (H_{orb}^*([M/G]; \mathbb{Q}), \circ).$$

is a ring isomorphism.

2.8. Chen-Ruan Cohomology. The Chen-Ruan cohomology is defined at the level of the quotient spaces M^g/G where the obstruction class is the Euler class of the orbundle $D_{g,h}/G$ over $M^{g,h}/G$ (see [CR04]). Namely

$$H_{CR}^*([M/G]) := \bigoplus_{g \in \mathcal{C}} H^*(M^g/G, \mathbb{Q})$$

with the product structure given by

$$\alpha \cdot \beta := e_{3*} (e_1^* \alpha \ e_2^* \beta \ eu(D_{g,h}/G)).$$

As the projection maps $\pi_g : M_G^g \rightarrow M^g/G$ induce isomorphisms in cohomology

$$\pi_g^* : H^*(M^g/G, \mathbb{Q}) \xrightarrow{\cong} H^*(M_G^g; \mathbb{Q})$$

(because the cohomology with rational coefficients of the fibers of π_g are acyclic), and they commute with pullbacks and pushforwards, we can conclude

Theorem 2.23. *The maps π_g induce an isomorphism of rings*

$$\pi^* : H_{CR}^*([M/G]) \rightarrow H_{orb}^*([M/G], \mathbb{Q}).$$

Proof. The isomorphism follows from the fact that

$$\pi_{g,h}^*(D_{g,h}/G) \cong \tau_{M^{g,h}} D_{g,h}.$$

\square

Composing the ring maps τ , Ch and π^* we obtain

Corollary 2.24. *The composition*

$$\pi^* \circ Ch \circ \tau : K_{st}^*([M/G]) \rightarrow H_{CR}^*([M/G])$$

is a ring homomorphism.

Example 2.25. The case of $\mathbb{CP}[p : 1 : \dots : 1]$

In the case of the weighted projective orbifold that we elaborated in section 2.4, we just need to disregard the coordinate of the cyclotomic extension of theorem 2.13; then we have that

$$K_{orb}^*([M/G]) \otimes \mathbb{Q} \cong \mathbb{Q}[u, g] / \langle (1-u)^{n+1}, g^p - (1-u)^n, g^{p+1} \rangle.$$

Applying the orbifold Chern character, which in this case is simply the Chern character map, we obtain the Chen-Ruan orbifold cohomology of the weighted projective space:

$$H_{orb}^*(\mathbb{CP}[p : 1 : \dots : 1]) \cong \mathbb{Q}[x, g] / \langle x^{n+1}, g^p - x^n, g^{p+1} \rangle,$$

where $\text{ch}(1-u)^n = x^n$.

3. STRINGY PRODUCT ON TWISTED ORBIFOLD K-THEORY

This chapter is devoted to study some properties of the stringy product on twisted orbifold K-theory for orbifolds of the type $[M/G]$ where G is a abelian finite group.

We will first recall the definition of the twisted orbifold K-theory and its stringy product following [ARZ]. Then we will show that the twistings used for the stringy product, that come from the inverse transgression, are all torsion. In the case that the twistings come from the cohomology of the group G , we will show a decomposition formula that is very suited to study the stringy product. This decomposition formula can be seen as a hybrid between the ring decomposition of equivariant K-theory (see [AR03, Thm 5.1]) and the group decomposition of twisted K-theory (see [AR03, Thm. 7.4]). Finally we will show an explicit calculation of the stringy product of twisted orbifold K-theory on which the twistings are non-trivial.

3.1. Stringy product on twisted orbifold K-theory. In this section we will review the definition of the twisted orbifold K-theory. For the details on the construction see [AR03, ARZ].

An element $\beta \in H_G^3(M; \mathbb{Z})$ defines ${}^\beta K_G^*(M)$, the twisted G -equivariant K-theory of M . If β comes from $H^3(BG; \mathbb{Z})$, then β defines a central extension of G by the circle

$$1 \rightarrow S^1 \rightarrow \tilde{G} \rightarrow G \rightarrow 0$$

and the twisted G -equivariant K-theory ${}^\beta K_G^*(M)$ becomes the group of \tilde{G} -equivariant vector bundles over M such that S^1 acts by multiplication on the fibers.

For $\alpha \in H_G^4(M; \mathbb{Z})$ the stringy product of the twisted orbifold K-theory ${}^\alpha K_{st}^*([M/G])$ is defined as the group

$${}^\alpha K_{st}^*([M/G]) := \bigoplus_{g \in G} {}^{\alpha_g} K_G^*(M^g) \times \{g\}$$

where $\alpha_g \in H_G^3(M^g; \mathbb{Z})$ is a twisting class that is the image of α under the inverse transgression map $\tau_g : H_G^4(M; \mathbb{Z}) \rightarrow H_G^3(M^g; \mathbb{Z})$ (see 3.1).

Let's denote the inclusions by

$$e_1 : M^{g,h} \rightarrow M^g \quad e_2 : M^{g,h} \rightarrow M^h \quad e_3 : M^{g,h} \rightarrow M^{gh}$$

and recall that $D_{g,h} \in K_G^*(M^{g,h})$ is the obstruction bundle of definition 2.2.

Definition 3.1. For the almost complex orbifold $[M/G]$, define the stringy product \star in the twisted orbifold K-theory ${}^\alpha K_{st}^*([M/G])$ as follows:

$$\begin{aligned} ({}^\alpha K_G^*(M^g) \times \{g\}) \times ({}^\alpha K_G^*(M^h) \times \{h\}) &\xrightarrow{\star} {}^\alpha K_G^*(M^{gh}) \times \{gh\} \\ ((F, g), (H, h)) &\mapsto (e_3!(e_1^* F \otimes e_2^* H \otimes \lambda_{-1}(D_{g,h})), gh). \end{aligned}$$

The proof of the fact that the \star operation makes ${}^\alpha K_{st}^*([M/G])$ into a ring can be found in [ARZ]. The product is well defined because $\alpha_g + \alpha_h = \alpha_{gh}$ and the associativity of \star follows from the properties of the obstruction bundles as in section 2.1.

3.2. Inverse transgression map. In this section we will show that the transgressed classes α_g are all torsion.

Let us recall the definition of the inverse transgression map for the orbifold $[M/G]$ when G is finite not necessarily abelian.

For $g \in G$ let $C(g)$ denote the centralizer of g in G . Consider the action of $C(g) \times \mathbb{Z}$ on M^g given by $(h, m) \cdot x := xhg^m$ and the homomorphism

$$\begin{aligned} \phi_g : C(g) \times \mathbb{Z} &\rightarrow G \\ (h, m) &\mapsto hg^m. \end{aligned}$$

Then the inclusion $i_g : M^g \rightarrow M$ becomes a ϕ_g -equivariant map and it induces a homomorphism

$$i_g^* : H_G^*(M; \mathbb{Z}) \rightarrow H_{C(g) \times \mathbb{Z}}^*(M^g; \mathbb{Z}).$$

As the group \mathbb{Z} acts trivially on M^g then we have that

$$H_{C(g) \times \mathbb{Z}}^*(M^g; \mathbb{Z}) \cong H^*(M^g \times_{C(g)} EC(g) \times B\mathbb{Z}; \mathbb{Z}),$$

and as $B\mathbb{Z} \simeq S^1$, via the Kunneth isomorphism we get that

$$H_{C(g) \times \mathbb{Z}}^*(M^g; \mathbb{Z}) \cong H_{C(g)}^*(M^g; \mathbb{Z}) \otimes_{\mathbb{Z}} H^*(S^1; \mathbb{Z}).$$

Therefore

$$i_g^* : H_G^p(M; \mathbb{Z}) \rightarrow H_{C(g)}^p(M^g; \mathbb{Z}) \oplus H_{C(g)}^{p-1}(M^g; \mathbb{Z})$$

and projecting on the second summand we get the inverse transgression map:

$$(3.1) \quad \tau_g : H_G^p(M; \mathbb{Z}) \rightarrow H_{C(g)}^{p-1}(M^g; \mathbb{Z}).$$

Lemma 3.2. *The transgressed classes are all torsion, i.e. for $\alpha \in H_G^p(M; \mathbb{Z})$ the inverse transgression $\alpha_g := \tau_g(\alpha) \in H_{C(g)}^{p-1}(M^g; \mathbb{Z})$ is a torsion class.*

Proof. Let $\langle g \rangle$ denote the cyclic group generated by g . Then the homomorphism ϕ_g factors through

$$C(g) \times \mathbb{Z} \rightarrow C(g) \times \langle g \rangle \rightarrow G \quad (h, m) \mapsto (h, g^m) \mapsto hg^m$$

and therefore i_g^* factors through

$$H_G^*(M; \mathbb{Z}) \rightarrow H^*(M^g \times_{C(g)} EC(g) \times B\langle g \rangle; \mathbb{Z}) \rightarrow H^*(M^g \times_{C(g)} EC(g) \times B\mathbb{Z}; \mathbb{Z}).$$

Now, as the cohomology of $B\langle g \rangle$ is all torsion, the Kunnetth isomorphism tells us that the torsion free part of $H^p(M^g \times_{C(g)} EC(g) \times B\langle g \rangle; \mathbb{Z})$ comes from

$$H_{C(g)}^p(M^g; \mathbb{Z}) \otimes_{\mathbb{Z}} H^0(B\langle g \rangle; \mathbb{Z}).$$

As the inverse transgression map does not factor through $H_{C(g)}^p(M^g; \mathbb{Z}) \otimes_{\mathbb{Z}} H^0(B\langle g \rangle; \mathbb{Z})$, then the transgressed classes must be torsion. \square

Example 3.3. Let's calculate the inverse transgression map for the group $G = (\mathbb{Z}/2)^3$; here we follow the analysis of lemma 5.2 in [ARZ].

Let's suppose first that the coefficients are in the field \mathbb{F}_2 of two elements. Then we have that

$$H^*(G; \mathbb{F}_2) \cong \mathbb{F}_2[x_1, x_2, x_3]$$

is the polynomial algebra on three generators. Take $g = (a_1, a_2, a_3) \in G$ and consider the maps

$$G \times \mathbb{Z} \rightarrow G \times \mathbb{Z}/2 \rightarrow G \quad (h, m) \mapsto (h, mg) \mapsto h + mg.$$

In cohomology we get

$$\begin{aligned} H^*(BG, \mathbb{F}_2) &\rightarrow H^*(BG \times B\mathbb{Z}/2; \mathbb{F}_2) \rightarrow H^*(BG \times B\mathbb{Z}; \mathbb{F}_2) \\ \mathbb{F}_2[x_1, x_2, x_3] &\rightarrow \mathbb{F}_2[x_1, x_2, x_3, w] \rightarrow \mathbb{F}_2[x_1, x_2, x_3, w]/(w^2) \\ x_i &\mapsto x_i + a_i w \mapsto x_i + a_i w \end{aligned}$$

and therefore the inverse transgression map with \mathbb{F}_2 coefficients is

$$\begin{aligned} \tau_g : \mathbb{F}_2[x_1, x_2, x_3] &\rightarrow \mathbb{F}_2[x_1, x_2, x_3] \\ x_1^{m_1} x_2^{m_2} x_3^{m_3} &\mapsto (a_1 m_1 x_1^{m_1-1} x_2^{m_2} x_3^{m_3} + a_2 m_2 x_1^{m_1} x_2^{m_2-1} x_3^{m_3} + a_3 m_3 x_1^{m_1} x_2^{m_2} x_3^{m_3-1}). \end{aligned}$$

For abelian 2-groups, the map $H^k(BG; \mathbb{Z}) \rightarrow H^k(BG; \mathbb{F}_2)$ is a monomorphism, and as $H^k(BG; \mathbb{Z}) \cong H^k(BG; \mathbb{Z}/4)$, then the long exact sequence induced by the extension $0 \rightarrow \mathbb{Z}/2 \rightarrow \mathbb{Z}/4 \rightarrow \mathbb{Z}/2 \rightarrow 0$ implies that

$$H^k(BG; \mathbb{Z}) \cong \ker(Sq^1 : H^k(BG; \mathbb{F}_2) \rightarrow H^{k+1}(BG; \mathbb{F}_2));$$

here Sq^1 is the Steenrod square operation that in this case is equivalent to the Bockstein map.

If we want to find elements in $H^4(BG; \mathbb{Z})$ that induce non trivial twistings via the inverse transgression map, one needs to find a homogenous polynomial $p(x_1, x_2, x_3) \in \mathbb{F}_2[x_1, x_2, x_3]$ of degree 4, such that $Sq^1(p) = 0$ and that $\tau_g(p) \neq 0$ for some $g \in G$. In [ARZ, Lemma 5.2] it is shown that the element

$$(3.2) \quad \alpha = Sq^1(x_1 x_2 x_3) = x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2$$

satisfies these properties. Its inverse transgression in $\tau_g : H^4(BG; \mathbb{Z}) \rightarrow H^3(BG, \mathbb{Z})$ for $g = (a_1, a_2, a_3)$ is equal to

$$(3.3) \quad \alpha_g = \tau_g(\alpha) = a_1(x_2^2 x_3 + x_2 x_3^2) + a_2(x_3^2 x_1 + x_3 x_1^2) + a_3(x_1^2 x_2 + x_1 x_2^2).$$

Let us calculate the double inverse transgression of α .

Lemma 3.4. *Let $g = (a_1, a_2, a_3)$ and $h = (b_1, b_2, b_3)$ be elements in $G = (\mathbb{Z}/2)^3$. The double inverse transgression of α is equal to:*

$$\begin{aligned} \tau_h \tau_g \alpha &= (a_2 b_3 + a_3 b_2) x_1^2 + (a_3 b_1 + a_1 b_3) x_2^2 + (a_1 b_2 + a_2 b_1) x_3^2 \\ &= [(a_1, a_2, a_3) \times (b_1, b_2, b_3)] \cdot (x_1^2, x_2^2, x_3^2). \end{aligned}$$

and therefore $\tau_h \tau_g \alpha \neq 0$ if and only if $g \neq h$ and $g \neq 0 \neq h$.

Proof. The formula follows from the fact that the inverse transgression $\tau_h(x_i^2 x_j)$ is $b_j x_i^2$. Now, the cross product $(a_1, a_2, a_3) \times (b_1, b_2, b_3)$ is equal to zero only when either $g = 0$ or $h = 0$, or $g = h$. \square

3.3. Decomposition formula for twisted orbifold K-theory. This section is devoted to show a decomposition formula for the twisted orbifold K-theory that is suited to study the stringy product; this formula is a simple generalization of [AR03, Thm. 7.4]. On what follows we will elaborate on the results of sections 5 and 6 of [AR03].

Let's assume that G is abelian and that $\beta : G \times G \rightarrow S^1$ is a cocycle representing the twisting class which correspond to the group extension $1 \rightarrow S^1 \rightarrow \tilde{G}_\beta \rightarrow G \rightarrow 0$. Therefore \tilde{G}_β is isomorphic to the group $G \times S^1$ with the product given by $(g_1, c_1)(g_2, c_2) = (g_1 g_2, \beta(g_1, g_2) c_1 c_2)$.

We will start by recalling some fact about projective representations that can be found on [AR03]. Recall that the group $R_\beta(G) := {}^\beta K_G^*(pt)$ of β -twisted representations can be understood as the subgroup of $R(\tilde{G}_\beta)$ generated by representations where S^1 acts by scalar multiplication.

To understand the group $R_\beta(G)$ we will restrict it to all the cyclic subgroups of G ; let's do this choosing a generator on each cyclic subgroup. Then, for g in G and $C = \langle g \rangle$ the cyclic group it generates, by restriction we have a map $R_\beta(G) \rightarrow R_{res(\beta)}(\langle g \rangle)$ where $res(\beta)$ is the restriction of β to the group $\langle g \rangle$. As $H^2(\langle g \rangle; S^1) = 0$ we have that $res(\beta)$ is cohomologous to zero, and therefore $R_{res(\beta)}(\langle g \rangle) \cong R(\langle g \rangle)$ the untwisted representation ring of $\langle g \rangle$.

Now, these restrictions to the cyclic subgroups are endowed with an action of the group G ; for this we need to make use of the inverse transgression of the cocycle β . The inverse transgression of β for $h \in G$ is a 1-cocycle, and therefore is given by a homomorphism $\tau_h \beta : G \rightarrow S^1$, where $g \mapsto \beta(h, g) \beta(g, h)^{-1}$ (see [LU02, Lemma 6.4.1]). Note that these homomorphisms also have the property that $\tau_h(g) \cdot \tau_k(g) = \tau_{hk}(g)$.

Having the inverse transgression map in mind, we can show now the natural action of G on $R_{res(\beta)}(\langle g \rangle)$. So, given a representation

$$\rho : \widetilde{\langle g \rangle}_\beta \rightarrow Aut(V),$$

for $(x, b) \in \widetilde{\langle g \rangle}_\beta$ and $z \in G$ we can define the representation

$$[z \cdot \rho] : \widetilde{\langle g \rangle}_\beta \rightarrow Aut(V)$$

such that

$$[z \cdot \rho](x, b) := \rho((z, 1)(x, b)(z, 1)^{-1}).$$

Note that this representation has the property that

$$(3.4) \quad [z \cdot \rho](x, b) = \tau_x(z) \rho(x, b)$$

and therefore as τ_x is a homomorphism one has that $[z_2 \cdot [z_1 \cdot \rho]] = [z_1 z_2 \cdot \rho]$. This shows that there is a natural action of G on $R_{res(\beta)}(\langle g \rangle)$.

We have that the class $res(\beta)$ is trivial in cohomology, therefore there is a function $\gamma : G \rightarrow S^1$ such that $\delta\gamma = \beta$. Then we can make a $res(\beta)$ -twisted representation of $\langle g \rangle$ into a representation of $\langle g \rangle$ in the following way. A $res(\beta)$ -twisted representation of $\langle g \rangle$ is given by a function $\psi : \langle g \rangle \rightarrow Aut(V)$ where $\psi(hk) = [res(\beta)(h, k)]\psi(h)\psi(k)$. But as $res(\beta)(h, k) = \gamma(k)\gamma(hk)^{-1}\gamma(h)$ we have that $\gamma(hk)\psi(hk) = \gamma(h)\psi(h)\gamma(k)\psi(k)$ and therefore $\gamma(\cdot)\psi(\cdot)$ becomes a $\langle g \rangle$ representation and we get the isomorphism

$$A_{\beta, g} : R_{res(\beta)}(\langle g \rangle) \cong R(\langle g \rangle); \quad \psi \mapsto \gamma\psi.$$

Now take ζ to be a $|g|$ -primitive root of unity, $\mathbb{Q}(\zeta)$ the cyclotomic field it generates and

$$\chi_g : R(\langle g \rangle) \rightarrow \mathbb{Q}(\zeta)$$

the trace map on the element g , i.e. $\chi_g \rho := \text{Trace}(\rho(g))$. Endow the group $R(\langle g \rangle)$ with the G action coming from the isomorphism $A_{\beta, g}$ and the cyclotomic field $\mathbb{Q}(\zeta)$ with the G action given by multiplication with $\tau_g \beta$, i.e. for $h \in G$ and $p(\zeta) \in \mathbb{Q}(\zeta)$ let $h \cdot p(\zeta) = \tau_g \beta(h)p(\zeta)$. In view of equation (3.4) the maps $A_{\beta, g}$ and χ_g become G -equivariant.

As the G action on $R(\langle g \rangle)$ and $\mathbb{Q}(\zeta)$ depends on β , we will label these groups by $R(\langle g \rangle)_\beta$ and $\mathbb{Q}(\zeta)_\beta$; the subindex β keeps track on how the G action is defined. Let's now show the result on which the calculation of the stringy product is based.

Lemma 3.5. *Let $\beta_1, \beta_2 : G \times G \rightarrow S^1$ be two 2-cocycles representing two twisting classes. Then the diagram*

$$\begin{array}{ccc} R_{res(\beta_1)}(\langle g \rangle) \times R_{res(\beta_2)}(\langle g \rangle) & \xrightarrow{\otimes} & R_{res(\beta_1 \beta_2)}(\langle g \rangle) \\ A_{\beta_1, g} \times A_{\beta_2, g} \downarrow & & \downarrow A_{\beta_1 \beta_2, g} \\ R(\langle g \rangle)_{\beta_1} \times R(\langle g \rangle)_{\beta_2} & \xrightarrow{\otimes} & R(\langle g \rangle)_{\beta_1 \beta_2} \\ \chi_g \times \chi_g \downarrow & & \downarrow \chi_g \\ \mathbb{Q}(\zeta)_{\beta_1} \times \mathbb{Q}(\zeta)_{\beta_2} & \xrightarrow{\cdot} & \mathbb{Q}(\zeta)_{\beta_1 \beta_2} \end{array}$$

is commutative and G -equivariant

Proof. Let $\psi_i : \langle g \rangle \rightarrow Aut(V_i)$ be an element in $R_{res(\beta_i)}(\langle g \rangle)$ and let $\gamma_i : G \rightarrow S^1$ be the function such that $\delta\gamma_i = \beta_i$. We have that $A_{\beta_i, g}(\psi_i) = \gamma_i \psi_i$ and therefore

$$A_{\beta_1, g}(\psi_1) \otimes A_{\beta_2, g}(\psi_2) = \gamma_1 \psi_1 \otimes \gamma_2 \psi_2 = \gamma_1 \gamma_2 (\psi_1 \otimes \psi_2) = A_{\beta_1 \beta_2, g}(\psi_1 \otimes \psi_2)$$

where the last equality follows from the fact that $\delta(\gamma_1 \gamma_2) = \beta_1 \beta_2$.

The commutativity of the second diagram follows from the fact that

$$\text{Trace}(A \otimes B) = \text{Trace}(A)\text{Trace}(B).$$

Now, for $h \in G$ we have that

$$(h \cdot \rho_1) \otimes (h \cdot \rho_2) = \tau_g \beta_1(h) \rho_1 \otimes \tau_g \beta_2(h) \rho_2 = \tau_g \beta_1 \beta_2(h) \rho_1 \otimes \rho_2 = h \cdot (\rho_1 \otimes \rho_2)$$

where the middle equality follows from the fact that τ_g is a homomorphism. This implies the equivariantness for the three horizontal lines. \square

The previous lemma shows how the restricted twisted representations behave once they are tensored, but we still need to show what is the appropriate decomposition for the twisted K-theory on which we can use lemma 3.5.

Let X be a G -CW complex and $\beta : G \times G \rightarrow S^1$ a 2-cocycle representing the twisting class. Denote X^g the fixed point set of g and consider the following homomorphisms

$${}^\beta K_G^*(X) \rightarrow {}^{res(\beta)} K_{\langle g \rangle}^*(X^g) \xrightarrow{\cong} K^*(X^g) \otimes R_{res(\beta)}(\langle g \rangle) \rightarrow K^*(X^g) \otimes \mathbb{Q}(\zeta)_\beta$$

where the first one is the restriction map, the second is the natural isomorphism given by the trivial action of g in X^g and the last one is given by the composition $\chi_g \circ A_{\beta, g}$. The composition of these homomorphisms has image in the invariants under the G action, then putting all these maps together for all the cyclic subgroups of G we get the following result

Theorem 3.6 (Decomposition for twisted K-theory). *Let X be a G -CW complex, G a finite group and $\beta : G \times G \rightarrow S^1$ a 2-cocycle. Let T be a subset of G such that any cyclic group of G is generated by only one element in T . Then we have a decomposition*

$${}^\beta K_G^*(X) \otimes \mathbb{Q} \cong \prod_{g \in T} (K^*(X^g) \otimes \mathbb{Q}(\zeta_{|g|})_\beta)^G$$

where $\zeta_{|g|}$ is a primitive $|g|$ -th root of unity.

Proof. The isomorphism follows directly from the decomposition formula of Adem and Ruan [AR03, Theorem 7.4].

Once checks first that it holds for $X = G/H$, and then one proceeds by induction on the number of G cells in X with the use of the Mayer-Vietoris exact sequence. \square

In view of lemma (3.5) we have that

Theorem 3.7. *The following diagram is commutative*

$$\begin{array}{ccc} {}^{\beta_1} K_G^*(X) \times {}^{\beta_2} K_G^*(X) & \xrightarrow{\otimes} & {}^{\beta_1 \beta_2} K_G^*(X) \\ \downarrow & & \downarrow \\ (K^*(X^g) \otimes \mathbb{Q}(\zeta_{|g|})_{\beta_1})^G \times (K^*(X^g) \otimes \mathbb{Q}(\zeta_{|g|})_{\beta_2})^G & \longrightarrow & (K^*(X^g) \otimes \mathbb{Q}(\zeta_{|g|})_{\beta_1 \beta_2})^G \end{array}$$

where the application of the bottom line takes the pair $(E_1 \otimes p_1(\zeta), E_2 \otimes p_2(\zeta))$ and maps it to $(E_1 \otimes E_2) \otimes (p_1(\zeta)p_2(\zeta))$.

Therefore the stringy product in twisted orbifold K-theory can be calculated with the use of the decomposition formula of theorem 3.6, and the fact that the stringy product structure decomposes as is shown in lemma 3.7.

Corollary 3.8. *In the case that the twistings β_1 and β_2 are zero we recover the ring isomorphism of [AR03, Thm 5.1]*

$$K_G^*(X) \otimes \mathbb{Q} \cong \prod_{g \in T} (K^*(X^g) \otimes \mathbb{Q}(\zeta_{|g|}))^G.$$

3.4. Examples. We finish this chapter by showing how the decomposition formula of theorem 3.6 is suited for calculating the stringy product in twisted orbifold K-theory. Let $G = (\mathbb{Z}/2)^3$ and $\alpha \in H^4(G, \mathbb{Z})$ as in formula (3.2), we will calculate ${}^\alpha K_{st}^*([*/G])$ and ${}^\alpha K_{st}^*([T^6/G])$.

Let us remark that the ring ${}^\alpha K_{st}^*([*/G])$ has already been calculated by A. N. Duman in [Dum] without tensoring with \mathbb{Q} by finding out the twisted representations and their products. We include here the calculation of this ring to make use of our decomposition formula and to clarify the calculation of ${}^\alpha K_{st}^*([T^6/G])$.

Example 3.9. ${}^\alpha K_{st}^*([*/G])$

As a group we have that

$${}^\alpha K_{st}^*([*/G]) = \bigoplus_{g \in G} {}^{\alpha_g} K_G^*(pt) = \bigoplus_{g \in G} R_{\alpha_g}(G)$$

where α_g is the inverse transgression of α in $H^3(G, \mathbb{Z})$ defined in formula (3.3).

By the decomposition formula we have that

$$R_{\alpha_g}(G) \otimes \mathbb{Q} \cong \prod_{h \in G} (\mathbb{Q}(\zeta_{|h|})_{\alpha_g})^G = \prod_{h \in G} (\mathbb{Q}_{\alpha_g})^G$$

where $T = G$ as all the elements generate a different cyclic group and $\mathbb{Q}(\zeta_{|h|}) = \mathbb{Q}$ because $\zeta_{|h|}$ is either 1 or -1 .

Recall that the action of k in $\mathbb{Q}(\zeta_{|h|})_{\alpha_g}$, for $k \in G$, is given by multiplication by $\tau_h \alpha_g(k)$, the inverse transgression of α_g evaluated in k . Therefore, by lemma 3.4 we have that

$$V_{g,h} =: (\mathbb{Q}(\zeta_{|h|})_{\alpha_g})^G = \begin{cases} \mathbb{Q} & \text{for } g = h, \text{ or } g = 0 \text{ or } h = 0 \\ 0 & \text{otherwise} \end{cases}$$

Therefore we have that for $g \neq 0$

$$R_{\alpha_g}(G) \otimes \mathbb{Q} \cong V_{g,1} \times V_{g,g} \cong \mathbb{Q} \times \mathbb{Q},$$

and as $\alpha_1 = \tau_1 \alpha = 0$ we have that

$$R_{\alpha_1}(G) \otimes \mathbb{Q} = R(G) \otimes \mathbb{Q} \cong \prod_{h \in G} \mathbb{Q}$$

where this last isomorphism is an isomorphism of rings. Therefore we have that the rank of ${}^\alpha K_{st}^*([*/G])$ is $2 \times 7 + 8 = 22$. We point out here that the rank of this ring was already calculated in [ARZ, Example 6.2].

The product structure is easy to determine following lemma 3.7. Let's denote by $1_{g,h} \in \prod_{h \in G} V_{g,h}$ the element which is 1 in the component of h and 0 on the other components. Then we have that

$$1_{g_1, h_1} \cdot 1_{g_2, h_2} = \begin{cases} 1_{1, h_1} & \text{if } 0 = g_1 g_2 \text{ and } h_1 = h_2 \\ 1_{g_1 g_2, h_1} & \text{if } h_1 = h_2 = g_1 g_2 \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

We finish this example by noting that the ring generated by $\bigoplus_{g \in G} V_{g,1}$ is isomorphic to the group ring $\mathbb{Q}[G]$. Therefore we have that both the group ring $\mathbb{Q}[G]$ and the representation ring $R(G) \otimes \mathbb{Q}$ are subrings of ${}^\alpha K_{st}^*([*/G])$; the last one because $R_{\alpha_1}(G) = R(G)$.

Example 3.10. ${}^\alpha K_{st}^*([T^6/G])$

Let's consider the torus $T^2 = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z})$ together with the $\mathbb{Z}/2$ action given by multiplication by -1 . The action fixes four points $\{0, \frac{1}{2}, \frac{1}{2}i, \frac{1}{2} + \frac{1}{2}i\}$ and we will denote this set by $4*$. The group $\mathbb{Z}/2$ acts on $K^*(T^2)$ by the trivial action on $K^0(T^2)$ and by the sign action on $K^1(T^2)$.

Then we can consider the group $G = (\mathbb{Z}/2)^3$ acting coordinate-wise on $M = T^6 = T^2 \times T^2 \times T^2$; that is, for $g = (a_1, a_2, a_3) \in G$ and $z = (z_1, z_2, z_3) \in T^6$, the action is $g \cdot z := ((-1)^{a_1} z_1, (-1)^{a_2} z_2, (-1)^{a_3} z_3)$.

On what follows we will explicitly calculate the rank of ${}^\alpha K_{st}^*([T^6/G])$ and we will show how the ring structure behaves for some chosen elements. We will not calculate how the ring structure behaves for all the cases because the rank of the twisted orbifold K-theory is too big.

Let's start by calculating the rank of ${}^\alpha K_{st}^*([T^6/G])$.

For $g \in G$, by theorem 3.6 we have that

$${}^{\alpha_g} K_G^*(M^g) \otimes \mathbb{Q} \cong \prod_{h \in G} (K^*(M^{g,h}) \otimes \mathbb{Q}_{h,\alpha_g})^G$$

where G acts on $\mathbb{Q}_{h,\alpha_g} = \mathbb{Q}$ multiplying by the double inverse transgression $\tau_h \tau_g \alpha = \tau_h \alpha_g$. Therefore for the elements $g \in \{(0,0,0), (1,0,0), (1,1,0), (1,1,1)\}$ we will elaborate some tables where we will calculate the double inverse transgressions $\tau_h \tau_g \alpha$, the fixed points $M^{g,h}$, and the invariant set $(K^*(M^{g,h}) \otimes \mathbb{Q}_{h,\alpha_g})^G$ together with its rank.

${}^{\alpha_g} K_G^*(M^g) \otimes \mathbb{Q}$ for $g = (1,0,0)$				
h	$\tau_h \tau_g \alpha$	$M^{g,h}$	$(K(M^{g,h}) \otimes \mathbb{Q}_{h,\alpha_g})^G$	rank
$(0,0,0)$	0	$4 * \times T^2 \times T^2$	$K^0(4*) \otimes K^0(T^2) \otimes K^0(T^2)$	16
$(1,0,0)$	0	$4 * \times T^2 \times T^2$	$K^0(4*) \otimes K^0(T^2) \otimes K^0(T^2)$	16
$(0,1,0)$	x_3^2	$4 * \times 4 * \times T^2$	$K^0(4*) \otimes K^0(4*) \otimes K^1(T^2)$	32
$(0,0,1)$	x_2^2	$4 * \times T^2 \times 4 *$	$K^0(4*) \otimes K^1(T^2) \otimes K^0(4*)$	32
$(1,1,0)$	x_3^2	$4 * \times 4 * \times T^2$	$K^0(4*) \otimes K^0(4*) \otimes K^1(T^2)$	32
$(0,1,1)$	$x_2^2 + x_3^2$	$4 * \times 4 * \times 4 *$	0	0
$(1,0,1)$	x_2^2	$4 * \times T^2 \times 4 *$	$K^0(4*) \otimes K^1(T^2) \otimes K^0(4*)$	32
$(1,1,1)$	$x_2^2 + x_3^2$	$4 * \times 4 * \times 4 *$	0	0
Total rank				160

${}^{\alpha_g} K_G^*(M^g) \otimes \mathbb{Q}$ for $g = (1,1,0)$				
h	$\tau_h \tau_g \alpha$	$M^{g,h}$	$(K(M^{g,h}) \otimes \mathbb{Q}_{h,\alpha_g})^G$	rank
$(0,0,0)$	0	$4 * \times 4 * \times T^2$	$K^0(4*) \otimes K^0(4*) \otimes K^0(T^2)$	32
$(1,0,0)$	x_3^2	$4 * \times 4 * \times T^2$	$K^0(4*) \otimes K^0(4*) \otimes K^1(T^2)$	32
$(0,1,0)$	x_3^2	$4 * \times 4 * \times T^2$	$K^0(4*) \otimes K^0(4*) \otimes K^1(T^2)$	32
$(0,0,1)$	$x_1^2 + x_2^2$	$4 * \times 4 * \times 4 *$	0	0
$(1,1,0)$	0	$4 * \times 4 * \times T^2$	$K^0(4*) \otimes K^0(4*) \otimes K^0(T^2)$	32
$(0,1,1)$	$x_1^2 + x_2^2 + x_3^2$	$4 * \times 4 * \times 4 *$	0	0
$(1,0,1)$	$x_1^2 + x_2^2 + x_3^2$	$4 * \times 4 * \times 4 *$	0	0
$(1,1,1)$	$x_1^2 + x_2^2$	$4 * \times 4 * \times 4 *$	0	0
Total rank				128

$\alpha_g K_G^*(M^g) \otimes \mathbb{Q}$ for $g = (1, 1, 1)$				
h	$\tau_h \tau_g \alpha$	$M^{g,h}$	$(K(M^{g,h}) \otimes \mathbb{Q}_{h, \alpha_g})^G$	rank
$(0, 0, 0)$	0	$4 * \times 4 * \times 4 *$	$K^0(4*) \otimes K^0(4*) \otimes K^0(4*)$	64
$(1, 0, 0)$	$x_2^2 + x_3^2$	$4 * \times 4 * \times 4 *$	0	0
$(0, 1, 0)$	$x_1^2 + x_3^2$	$4 * \times 4 * \times 4 *$	0	0
$(0, 0, 1)$	$x_1^2 + x_2^2$	$4 * \times 4 * \times 4 *$	0	0
$(1, 1, 0)$	$x_1^2 + x_2^2$	$4 * \times 4 * \times 4 *$	0	0
$(0, 1, 1)$	$x_2^2 + x_3^2$	$4 * \times 4 * \times 4 *$	0	0
$(1, 0, 1)$	$x_1^2 + x_3^2$	$4 * \times 4 * \times 4 *$	0	0
$(1, 1, 1)$	0	$4 * \times 4 * \times 4 *$	$K^0(4*) \otimes K^0(4*) \otimes K^0(4*)$	64
Total rank				128

For $g = (0, 0, 0)$ the inverse transgression $\tau_g \alpha = 0$ and therefore we just need to calculate the invariants for each h

$K_G^*(M^g) \otimes \mathbb{Q}$ for $g = (0, 0, 0)$			
h	$M^{g,h}$	$(K(M^{g,h}) \otimes \mathbb{Q})^G$	rank
$(0, 0, 0)$	$T^2 \times T^2 \times T^2$	$K^0(T^2) \otimes K^0(T^2) \otimes K^0(T^2)$	8
$(1, 0, 0)$	$4 * \times T^2 \times T^2$	$K^0(4*) \otimes K^0(T^2) \otimes K^0(T^2)$	16
$(0, 1, 0)$	$T^2 \times 4 * \times T^2$	$K^0(T^2) \otimes K^0(4*) \otimes K^0(T^2)$	16
$(0, 0, 1)$	$T^2 \times T^2 \times 4 *$	$K^0(T^2) \otimes K^0(T^2) \otimes K^0(4*)$	16
$(1, 1, 0)$	$4 * \times 4 * \times T^2$	$K^0(4*) \otimes K^0(4*) \otimes K^0(T^2)$	32
$(0, 1, 1)$	$T^2 \times 4 * \times 4 *$	$K^0(T^2) \otimes K^0(4*) \otimes K^0(4*)$	32
$(1, 0, 1)$	$4 * \times T^2 \times 4 *$	$K^0(4*) \otimes K^0(T^2) \otimes K^0(4*)$	32
$(1, 1, 1)$	$4 * \times 4 * \times 4 *$	$K^0(4*) \otimes K^1(4*) \otimes K^0(4*)$	64
Total rank			216

As the groups $\alpha_g K_G^*(M^g)$ are isomorphic for $g \in \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ and for $g \in \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$, then we can conclude that

$$\text{rank } \alpha K_{st}^*([T^6/G]) = 216 + 3 \times 160 + 3 \times 128 + 128 = 1208.$$

The stringy product now is easy to calculate. We just need to find the obstruction bundle and the pushforward maps.

First note that all the obstruction bundles are trivial. This is because the eigenvalues of the action of any element in G is 1 or -1 , and therefore the $a_j + b_j$ of definition 2.2 are always smaller or equal than 1.

The pushforwards of the stringy product can be calculated by understanding the $\mathbb{Z}/2$ -equivariant pushforward of the inclusion $4* \rightarrow T^2$. Let's denote by $\eta \in K_{\mathbb{Z}/2}^0(T^2)$ the trivial 1-dimensional bundle whose action of $\mathbb{Z}/2$ is given by multiplication by -1 composed with the geometrical action, and $H \in K_{\mathbb{Z}/2}^0(T^2)$ be the pullback of the Hopf bundle under the degree 1 map $T^2 \rightarrow S^2$ together with the pullback action of $\mathbb{Z}/2$. If we consider only one point $i : * \rightarrow T^2$ the pushforward map is

$$i_! : R(\mathbb{Z}/2) \rightarrow K_{\mathbb{Z}/2}^0(T^2), \quad i_!(1) = 1 - H \otimes \eta$$

and therefore $i^* i_!(1) = 1 - \xi$ where ξ is the sign representation.

For $E \in \alpha_g K_G^*(M^g)$ and $F \in \alpha_k K_G^*(M^k)$, by definition 3.1, their stringy product is

$$E \star F := e_{3!}(e_1^* E \otimes e_2^* F) \in \alpha_{gk} K_G^*(M^{gk}).$$

By the decomposition formula we have that $E \mapsto \prod_h E_h$ and $F \mapsto \prod_h F_h$ and by theorem 3.7 we have that $(e_1^* E \otimes e_2^* F)_h \cong e_1^* E_h \otimes e_2^* F_h$. The only part where one needs to be careful is with the behavior of the pushforward map

$$e_{3!} : {}^{\alpha_{gk}} K_G^*(M^{g,k}) \rightarrow {}^{\alpha_{gk}} K_G^*(M^{gk})$$

under the decomposition formula.

As there would be too many cases to consider, we will only show how to calculate the stringy product for $g = (1, 0, 0)$, $k = (1, 1, 0)$ and $gk = (0, 1, 0)$. For the other cases one can imitate the following procedure.

First let's calculate ${}^{\alpha_{gk}} K_G^*(M^{g,k})$ in order to know which components are non zero in its decomposition:

${}^{\alpha_{gk}} K_G^*(M^{g,k}) \otimes \mathbb{Q}$ for $g = (1, 0, 0)$ and $k = (1, 1, 0)$			
h	$\tau_h \tau_{gk} \alpha$	$M^{g,k,h}$	$(K(M^{g,k,h}) \otimes \mathbb{Q}_{h, \alpha_{gk}})^G$
$(0, 0, 0)$	0	$4 * \times 4 * \times T^2$	$K^0(4*) \otimes K^0(4*) \otimes K^0(T^2)$
$(1, 0, 0)$	x_3^2	$4 * \times 4 * \times T^2$	$K^0(4*) \otimes K^0(4*) \otimes K^1(T^2)$
$(0, 1, 0)$	0	$4 * \times 4 * \times T^2$	$K^0(4*) \otimes K^0(4*) \otimes K^0(T^2)$
$(0, 0, 1)$	x_1^2	$4 * \times 4 * \times 4*$	0
$(1, 1, 0)$	x_3^2	$4 * \times 4 * \times T^2$	$K^0(4*) \otimes K^0(4*) \otimes K^1(T^2)$
$(0, 1, 1)$	x_1^2	$4 * \times 4 * \times 4*$	0
$(1, 0, 1)$	$x_1^2 + x_3^2$	$4 * \times 4 * \times 4*$	0
$(1, 1, 1)$	$x_1^2 + x_3^2$	$4 * \times 4 * \times 4*$	0

Now let's see how the pushforward

$${}^{\alpha_{gk}} K_G^*(M^{g,k}) \xrightarrow{e_{3!}} {}^{\alpha_{gk}} K_G^*(M^{gk})$$

behaves on each of the nonzero components (on the left we have $(K^*(M^{g,k,h}) \otimes \mathbb{Q}_{h, \alpha_{gk}})^G$ and on the right $(K^*(M^{gk,h}) \otimes \mathbb{Q}_{h, \alpha_{gk}})^G$).

For $h = (0, 0, 0)$

$$K^0(4*) \otimes K^0(4*) \otimes K^0(T^2) \xrightarrow{(1-H) \otimes 1 \otimes 1} K^0(T^2) \otimes K^0(4*) \otimes K^0(T^2)$$

where the map is given by multiplying with the element $(1 - H) \otimes 1 \otimes 1$ (the Thom class of the inclusion).

For $h = (1, 0, 0)$

$$K^0(4*) \otimes K^0(4*) \otimes K^0(T^2) \xrightarrow{2 \otimes 1 \otimes 1} K^0(4*) \otimes K^0(4*) \otimes K^0(T^2)$$

because $(1 - H \otimes \eta) \otimes 1 \otimes 1$ restricted to $4 * \times 4 * \times T^2$ is $(1 - \xi) \otimes 1 \otimes 1$, and its character on $h = (1, 0, 0)$ is $(2 \otimes 1 \otimes 1)$.

For $h = (0, 1, 0)$

$$K^0(4*) \otimes K^0(4*) \otimes K^0(T^2) \xrightarrow{(1-H) \otimes 1 \otimes 1} K^0(T^2) \otimes K^0(4*) \otimes K^0(T^2)$$

because the character of $(1 - H \otimes \eta) \otimes 1 \otimes 1$ in $h = (0, 1, 0)$ is $1 - H \otimes 1 \otimes 1$.

For $h = (1, 1, 0)$

$$K^0(4*) \otimes K^0(4*) \otimes K^0(T^2) \xrightarrow{2 \otimes 1 \otimes 1} K^0(4*) \otimes K^0(4*) \otimes K^0(T^2)$$

because the character of $(1 - \xi) \otimes 1 \otimes 1$ in $h = (1, 1, 0)$ is $2 \otimes 1 \otimes 1$.

The decomposition for the stringy product $E \star F$ can be now easily calculated. Write $E_h := (E_h^1, E_h^2, E_h^3)$ and $F_h = (F_h^1, F_h^2, F_h^3)$ in terms of Kunneth decomposition as in the tables above. For example, for $h = (1, 1, 0)$ we have that $(E_h^1, E_h^2, E_h^3) \in K^0(4*) \otimes K^0(4*) \otimes K^1(T^2)$. If we call the inclusion $j : 4* \rightarrow T^2$, then we have that

$$(E \star F)_h = \begin{cases} ((1-H) \otimes E_h^1 \otimes F_h^1) \otimes (j^* E_h^2 \otimes F_h^2) \otimes (E_h^3 \otimes F_h^3) & \text{for } h = (0, 0, 0) \\ (2E_h^1 \otimes F_h^1) \otimes (j^* E_h^2 \otimes F_h^2) \otimes (E_h^3 \otimes F_h^3) & \text{for } h = (1, 0, 0) \\ ((1-H) \otimes E_h^1 \otimes F_h^1) \otimes (E_h^2 \otimes F_h^2) \otimes (E_h^3 \otimes F_h^3) & \text{for } h = (0, 1, 0) \\ (2E_h^1 \otimes F_h^1) \otimes (E_h^2 \otimes F_h^2) \otimes (E_h^3 \otimes F_h^3) & \text{for } h = (1, 1, 0) \\ 0 & \text{otherwise} \end{cases}$$

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